Probability

Counting Principles

Suppose it is necessary to find the number of ways a group of items can be arranged. This is a counting problem.

Example

There are 10 people moving into a neighborhood. In how many different ways can those 10 people be arranged into 10 different houses?

Number the houses from one to ten.

House 1 House 2 House 3 House 4 House 5 House 6

House 7 House 8 House 9 House 10

Consider how many different people are available to move into the first house?

Since there are 10 people moving in, then 10 different people could move into this house.

House 1 House 2 House 3 House 4 House 5 House 6

10

House 7 House 8 House 9 House 10

Now, one person has moved into the first house, how many different people are available to move into the second house?

Since one person moved into the first house there are now 10-1 = 9 people available to move into the second house.

House 1	House 2	House 3	House 4	House 5	House 6
10	9				

House 7	House 8	House 9	House 10

One person moved into the first house and one person moved into the second house, how many people are available to move into the third house?

10 – 2 = 8

House 1	House 2	House 3	House 4	House 5	House 6
10	9	8			

House 7	House 8	House 9	House 10

This is repeated till all the houses are full

House 1	House 2	House 3	House 4	House 5	House 6
10	9	8	7	6	5

House 7	House 8	House 9	House 10
4	3	2	1

To find the number of possible arrangements of the 10 people into the 10 houses multiply the number of people that could move into each one.

Number of arrangements $=10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=3,628,800$

What if there had been 200 houses, writing out the multiplication could take a while. There is a mathematical notation called factorial which is useful.

Factorial

$$n!=n(n-1)(n-2)(n-3)...(1)$$

The number of arrangements of 10 people into 10 houses is

$$10!==10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=3,628,800$$

This is a way to find the number of ways of arranging items when the number of places to put them is the same as the number of items. What if there are more items than places to put them?

There are two options, **Permutations** and **Combinations**.

The number of **permutations** (ordered arrangements) of n distinct objects taken r at a time is:

$$P_{n,r}=\frac{n!}{(n-r)!}$$

Where n and r are whole numbers and

$$n \geq r$$

Example: Compute the number of possible ordered seating arrangements for 10 flute players into 8 seats?

$$n=10 \quad r=8$$

$$P_{8,5}=\frac{10!}{(10-8)!}=\frac{10!}{2!}=\frac{3628800}{2}=1,814,400$$

In permutations the order of occurrence is important.

What if the order is not important? Then use combinations.

The number of **combinations** of n objects taken r at a time is:

$$C_{n.r}=\frac{n!}{r!(n-r)!}$$

where n and ra are whole numbers and

$$n \geq r$$

Example

In a political science class, a student is assigned to read 7 books from a list of 20. How many different groups of 7 are available from the list of 20?

The order in which the books are read does not matter on that the books are read. This means that the combination formula should be used.

$$n = 20 \qquad r = 7$$
$$C_{n.r} = \frac{n!}{r!(n-r)!}$$
$$C_{20,7} = \frac{20!}{7!(20-7)!} = \frac{20!}{7!(13!)} = 77520$$

There are 77,520 different combinations of books that the student can read.

Probability Calculations

The probability of an outcome of a random phenomenon is the proportion of times the outcome would occur in a very long sequence of repetitions of the experiment.

Example (Discrete)

Suppose you role a six sided fair die, what is the probability of rolling an even number?

$$P(\text{rolling an even number}) = \frac{\text{number of ways to roll and even number}}{\text{number of possible outcomes}}$$
$$P(\text{rolling an even number}) = \frac{3}{6} = \frac{1}{2} = .5$$

The number of ways to roll and even number is the number of even numbers on the die.

2, 4, 6 are the even numbers, so there are three

The number of possible outcomes is the number of sides on the die, which is 6

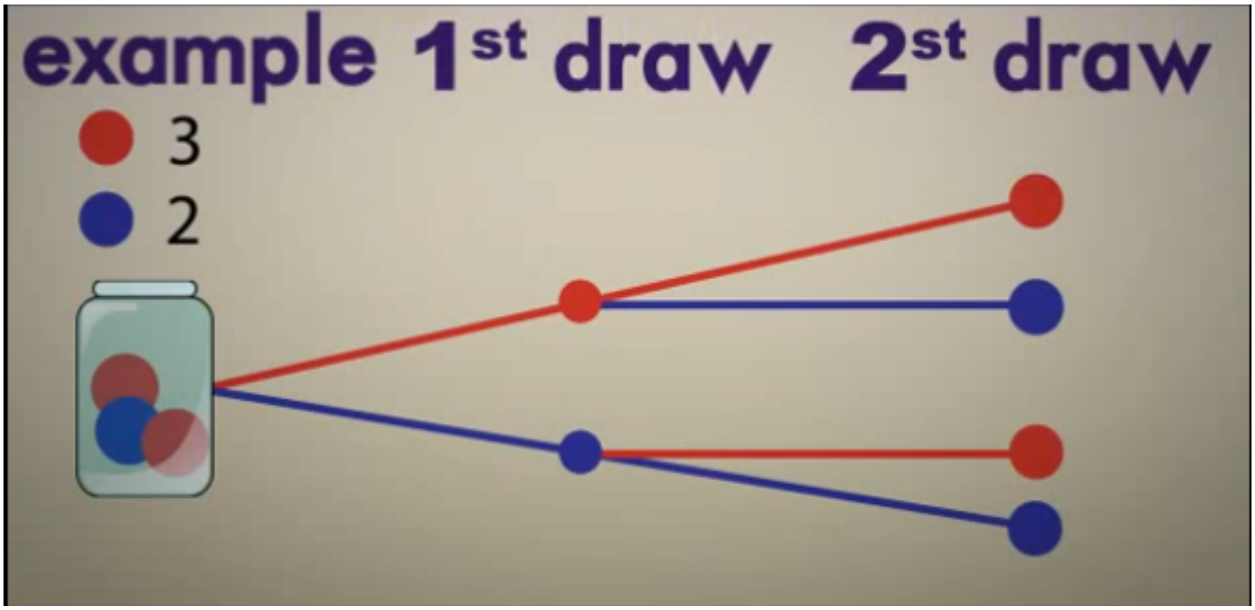
Conditional Probability

Conditional probabilities become important when a previous event has an effect on a subsequent effect.

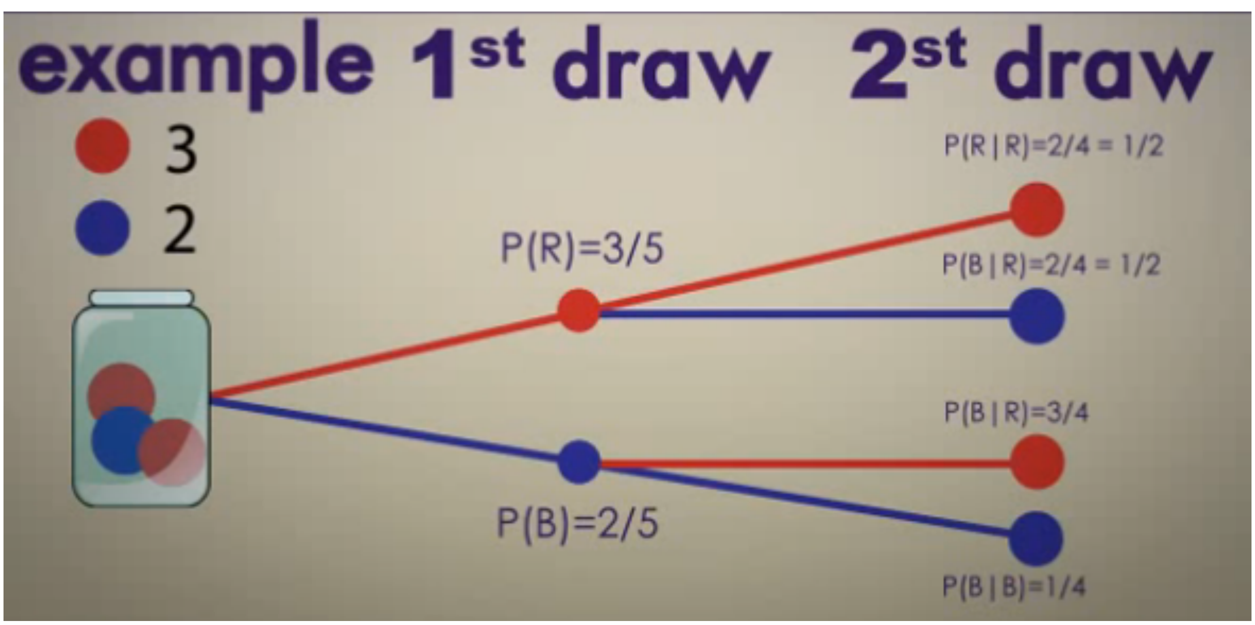
Example

There is a jar with 5 balls inside. There are three red balls and two blue balls. You draw one ball note the color, then draw a second ball without replacing the first. Note the color of the second ball. Find all the possible outcomes and the probabilities of all the possible outcomes.

An easy way to see all the possible outcomes is to draw a tree diagram.



On the second draw from the jar, the probabilities change based on what was taken out of the Jar in the first jar. This is a conditional probability.



Binomial Probability

To discuss binomial probabilities we must first understand what binomial experiments are. A binomial experiment has the following traits.

1. There are n observations (called Trials)
2. The observations are independent
3. Each observation results in only one of two possible outcomes – called success and failure.
4. Probability of success, p, is the same for all trials.
5. We are interested in the total number of successes in n trials

Some important notation

P(S) = p

P(F) = 1- p = q

n = number of trials

x = number of successes in n trials

P(x) = probability of getting exactly x successes in n trials

$$P\left(X = x\right) = C_{n,x}p^{n}q^{n-x} \text{ where } x = 0,1,2,\ldots,n$$

Example

At a certain college 60% of the students are male and 40% are female. Suppose 4 students are selected at random from the student body. Let X= number of females among the four. Find the probability that 2 of the four students are female?

We know this is a binomial experiment because there are exactly two possible outcomes, male or female.

x=number of females in the random selection.

n=sample size = 4

$$P(X=2)=C_{4,2}(.6)^4(.4)^{4-2}$$
$$P(X=2)=\frac{4!}{2!(4-2)!}(.6)^4(.4)^2$$
$$P(X=2)=6(.1296)(.16)$$
$$P(X=2)=.124416\approx .124$$

The probability there being two females in the random sample of 4 is approximately .124

What is we wanted to know the probability of there being less than two female students in the random sample of four.

To find this probability add up the probability of there being 0 females, 1 female.

$$P(X<2)=P(X=0)+P(X=1)$$
$$P(X<2)=C_{4,0}(.6)^4(.4)^{4-0}+C_{4,1}(.6)^4(.4)^{4-1}$$
$$P(X<2)=\frac{4!}{0!(4-0)!}(.6)^4(.4)^4+\frac{4!}{1!(4-1)!}(.6)^4(.4)^3$$
$$P(X<2)=1(.1296)(.0256)+4(.1296)(.064)$$
$$P(X<2)=.00331776+.0331776$$
$$P(X<2)=.0364954$$

The probability of there being less than two females in sample is approximately .0365.